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IMPACT OF A CONE ON A DEFORMABLE STRING*

D.G. AGALAROV, B.R. NURIEV and Kh.A. RAKHMATULIN

The motion of a flexible deformable string and the process of propagation of the stress waves through it, following a transverse impact of a cone, are investigated. Equations of motion are derived and an analytic solution of a self-similar problem is obtained for an arbitrary, single-valued stress-deformation relationship. Possible wave patterns of the string motion are studied. It is shown that in the case of complete encirclement the motion of a string with a single kink is impossible, and a new wave pattern of motion caused by a transverse impact is obtained for the first time. An established wave system with two kinks is used to derive the conditions at the strong shock waves. A numerical algorithm is given for a scheme with linear strengthening and the deformation versus impact rate graphs are constructed.

The theory of transverse impact on elastic strings when the deviations from the initial rectilinear shape are large, has been dealt with in /1,2/. The theory of plane motion of a string is well elucidated in /3−6/.

1. Equations of motion.

Let a cone execute a transverse impact at constant velocity v_0 on an infinte, flexible rectilinear string, with the cone axis passing across the line of initial position of the string, and the string slipping off the apex of the cone at the time of initial contact. Inverting the problem, we shall consider the motion of the string along the cone surface with transverse velocity v_0 at infinity.

We use, as the independent variables, the time t and the Lagrangian coordinate s_0, the latter denoting the distance of the particle in question from some fixed point at the initial instant. We shall define the positions of the particles moving along the cone surface in terms of the radius $r(s_0, t)$ of a circle passing through the point in question, (s_0, t), and central angle $\theta(s_0, t)$ of this circle counted from some fixed axial plane. In the absence of the frictional and mass forces, the equation of motion of the string on the cone surface has the form

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{T}}{\partial s_0} + \frac{\mathbf{P}}{F_0} \tag{1.1}$$

where $\mathbf{v}(s_0, t)$ is the velocity of the string particles, ρ_0 is the initial density, $\mathbf{T} = \sigma\boldsymbol{\tau}$ is the tension, $\sigma(e)$ is the stress referred to the initial plane of transverse cross section, $\boldsymbol{\tau}(s_0, t)$ is the unit vector tangent to the string at the point (s_0, t), $e(s_0, t)$ is the relative elongation, $\mathbf{P}(s_0, t)/F_0$ is the normal reaction of the cone and F_0 is the initial plane of transverse section of the string.

The string moving along the cone surface is flexible, therefore the following relations hold:

$$\mathbf{v} = \mathbf{i} r \frac{\partial \theta}{\partial t} + \mathbf{j} \frac{1}{\sin\alpha} \frac{\partial r}{\partial t}, \quad \mathbf{T} = \mathbf{i}\sigma\cos\varphi + \mathbf{j}\sigma\sin\varphi \tag{1.2}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the circumference and the generatrix of the cone, respectively, passing through the point (s_0, t), α is the semiangle of the cone and $\varphi(s_0, t)$ is the angle between the string and the circle on the cone surface at the point (s_0, t).

Calculating the derivatives of the unit vectors $\mathbf{i}(s_0, t)$ and $\mathbf{j}(s_0, t)$, we obtain

$$\frac{\partial \mathbf{i}}{\partial t} = \mathbf{n}^\circ \frac{\partial \theta}{\partial t}, \quad \frac{\partial \mathbf{i}}{\partial s_0} = \mathbf{n}^\circ \frac{\partial \theta}{\partial s_0}$$
$$\frac{\partial \mathbf{j}}{\partial t} = \mathbf{i} \frac{\partial \theta}{\partial t} \sin\alpha, \quad \frac{\partial \mathbf{j}}{\partial s_0} = \mathbf{i} \frac{\partial \theta}{\partial s_0} \sin\alpha \tag{1.3}$$

where $\mathbf{n}^\circ(s_0, t)$ is a unit vector originating at (s_0, t) and directed along the radius towards the center of the circle passing through the point (s_0, t).

Using the projections on the directions $\mathbf{i}$ and $\mathbf{j}$ we obtain from (1.1), with (1.2) and (1.3) taken into account,

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$$\rho_0\left(r\frac{\partial^2\theta}{\partial t^2}+2\frac{\partial\theta}{\partial t}\frac{\partial r}{\partial t}\right)=\frac{\partial}{\partial s_0}(\sigma\cos\varphi)+\sigma\sin\varphi\frac{\partial\theta}{\partial s_0}\sin\alpha \tag{1.4}$$
$$\rho_0\left[\frac{1}{\sin\alpha}\frac{\partial^2 r}{\partial t^2}-r\sin\alpha\left(\frac{\partial\theta}{\partial t}\right)^2\right]=\frac{\partial}{\partial s_0}(\sigma\sin\varphi)-\sigma\cos\varphi\frac{\partial\theta}{\partial s_0}\sin\alpha$$

We have the following kinematic conditions:

$$r\frac{\partial\theta}{\partial s_0}=(1+e)\cos\varphi,\qquad \frac{1}{\sin\alpha}\frac{\partial r}{\partial s_0}=(1+e)\sin\varphi \tag{1.5}$$

Equations (1.4) and (1.5) and the equations of state

$$\sigma=\sigma(e) \tag{1.6}$$

together represent a closed system of nonlinear equations in partial derivatives, from which we must obtain $r(s_0,t),\theta(s_0,t),\varphi(s_0,t),e(s_0,t)$. After this we find the pressure $|\mathbf{P}|=PF_0/(1+e)$ exerted by the cone, by projecting the equation (1.1) in the direction of the outward normal to the cone surface

$$-\rho_0 r\left(\frac{\partial\theta}{\partial t}\right)^2\cos\alpha=P-\sigma\cos\varphi\frac{\partial\theta}{\partial s_0}\cos\alpha \tag{1.7}$$

2. Solution of the problem. The system of equations (1.4), (1.5) and (1.6) implies that in the case of a transverse impact with constant velocity, the problem is self-similar.

Let us introduce the dimensionless variables

$$R=\frac{r}{v_0t},\quad x=\frac{s_0}{v_0t} \tag{2.1}$$

According to the theory of similarity the functions R,θ,φ,e sought must depend only on x. Consequently, we can write the system (1.3), (1.4) in the form (a prime denotes differentiation with respect to x)

$$x^2(R\theta''+2R'\theta')=\frac{d}{dx}(\sigma_0\cos\varphi)+\theta'\sin\alpha\sigma_0\sin\varphi,\quad \sigma_0=\frac{\sigma}{\rho_0v_0^2} \tag{2.2}$$
$$x^2\left[\frac{1}{\sin\alpha}R''-R\sin\alpha(\theta')^2\right]=\frac{d}{dx}(\sigma_0\sin\varphi)-\theta'\sigma_0\sin\alpha\cos\varphi$$

$$R\theta'=(1+e)\cos\varphi,\quad \frac{1}{\sin\alpha}R'=(1+e)\sin\varphi \tag{2.3}$$

Equation (1.7) now becomes

$$-x^2R\theta'^2\cos\alpha=P_0-\theta'\sigma_0\cos\alpha\cos\varphi \tag{2.4}$$

In addition to the equations (2.2)—(2.4), we have the equation of state (1.6).

The system (2.2), (2.4) with (2.3) taken into account yields, after certain manipulations,

$$\left(\frac{\sigma_0}{1+e}-x^2\right)(\varphi'-\theta'\sin\alpha)=0 \tag{2.5}$$
$$\left(\frac{d\sigma_0}{de}-x^2\right)\frac{de}{dx}=0,\quad P_0=\left(\frac{\sigma_0}{1+e}-x^2\right)\theta'\cos\alpha\cos\varphi$$

The last equation shows that $\sigma_0/(1+e)-x^2\neq 0$, since the force of reaction at the cone surface $\mathbf{P}\neq 0$ and hence $P_0\neq 0$. Therefore we obtain the following expression, from the first equation of (2.5), for an arbitrary relation $\sigma=\sigma(e)$:

$$\varphi=\theta\sin\alpha+C_1 \tag{2.6}$$

Let us divide the second equation of (2.3) by the first, and integrate with respect to x. Taking into account (2.6) we obtain

$$R=C_2/\cos\varphi \tag{2.7}$$

where C_1 and C_2 are integration constants to be determined.

Under the linearly elastic deformations $(\sigma=Ee)$ and nonlinear relations between the stress and deformation, we find for $d^2\sigma/de^2>0$, $d\sigma_0/de>\sigma_0/(1+e)$, with the condition $x^2-\sigma_0/(1+e)<0$ taken into account (since $P_0>0$), that

$$x^2-d\sigma_0/de<0$$

Consequently, from the second equation of (2.5) we obtain, for an arbitrary relationship $\sigma=\sigma(e)$ when $d^2\sigma/de^2\geqslant 0$,

$$e = \text{const.} \tag{2.8}$$

In the case of relations $\sigma = \sigma(e)$ for which $d^2\sigma/de^2 < 0$, the second equation of (2.5) yields the integral

$$x^2 - d\sigma_0/de = 0 \tag{2.9}$$

The solution (2.9) which is used to find $e = e(x)$ for the given relation $\sigma = \sigma(e)$ shows, that in the case of the relations $\sigma = \sigma(e)$ ensuring that the wave process is continuous, a Riemann wave propagates along the string with the region of constant deformations situated behind the Riemann wave. The form of the string described by the formula (2.6) is in this case three-dimensional.

From (2.3), with (2.6) and (2.7) taken into account, we obtain the last integral of the problem

$$\operatorname{tg}\varphi = \frac{\sin\alpha}{C_2}\int(1+e)\,dx + C_3 \tag{2.10}$$

In the case of a straight impact we have $\theta = 0$ and $\varphi = 0$ at $x = 0$. Consequently the constants C_1 and C_3 vanish and the formulas (2.6) and (2.10) assume the form

$$\varphi = \theta\sin\alpha, \quad \operatorname{tg}\varphi = \frac{\sin\alpha}{C_2}\int_0^x(1+e)\,dx \tag{2.11}$$

In the region of purely longitudinal motions we have the following one-dimensional equation:

$$\rho_0\frac{\partial^2 w}{\partial t^2} = \frac{\partial\sigma}{\partial s_0}, \quad e = \frac{\partial w}{\partial s_0}, \quad \sigma = \sigma(e) \tag{2.12}$$

where $w(s_0, t)$ denotes the displacement of the string particles in the region of purely longitudinal motions. It can be shown that when $d^2\sigma/de^2 < 0$, the equation (2.13) has a solution

$$x^2 - \frac{d\sigma_0}{de} = 0, \quad w_t = -\int_0^e\sqrt{\frac{1}{\rho_0}\frac{d\sigma}{de}}\,de \tag{2.13}$$

when $d^2\sigma/de^2 \geqslant 0$, (2.13) yields

$$e = \text{const.} \tag{2.14}$$

The boundary values of the deformation and rate of motion of the particles in the region of purely longitudinal motions are found from the conditions at the strong shock wave formed at the point at which the transverse motion becomes longitudinal. The constant C_2 appearing in the formulas (2.7) and (2.11) must be determined from the conditions at the point $\theta = \pi/2$.

3. Possible wave motion patterns. Let us write the equations of conservation of momentum at the strong shock wave A (Fig.1), taking into account the presence of a concentrated force Q at the point of contact with the cone

$$\begin{gathered}\rho_0(b - w_t)\,r\,\frac{\partial\theta}{\partial t} = -\sigma\cos\varphi\,(1+e_1)\\ \rho_0(b - w_t)\left(\frac{1}{\sin\alpha}\frac{\partial r}{\partial t} - v_0\cos\alpha - w_t\sin\alpha\right) = (\sigma_1\sin\alpha - \sigma\sin\varphi)(1+e_1)\\ \rho_0(b - w_t)(v_0\sin\alpha - w_t\cos\alpha) = (\sigma_1\cos\alpha + Q)(1+e_1)\end{gathered} \tag{3.1}$$

Here σ_1, e_1 and w_t, are respectively, the stress, deformation and velocity of the particles in the region of purely longitudinal motions, and b is the velocity of the strong shock wave.

The kinematic considerations yield

$$\begin{gathered}\frac{v_0}{\cos\alpha} - \frac{1}{\sin\alpha}\frac{\partial r}{\partial t} = (b - w_t)\frac{1+e}{1+e_1}\sin\varphi\\ -r\frac{\partial\theta}{\partial t} = (b - w_t)\frac{1+e}{1+e_1}\cos\varphi, \quad b = v_0\operatorname{tg}\alpha\end{gathered} \tag{3.2}$$

If $\varphi \neq \pi/2$, then the first two equations of (3.1) and (3.2) yield

$$\rho_0(b - w_t)^2 = \sigma_1(1+e_1) \tag{3.3}$$

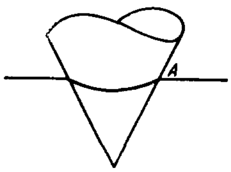

Fig.1

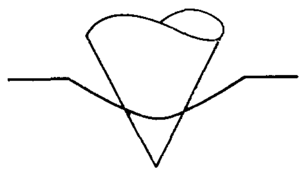

Fig.2

Substituting (3.3) into the last equation of (3.1) we obtain $Q = 0$, and this contradicts the assumption that a concentrated force is present. From this we deduce that a region of transverse motions exists in which the string is not in contact with the cone (Fig.2). This represents one of the possible patterns of motion realized when the velocity of the shock is small

$$v_0 \operatorname{tg} \alpha \leqslant \sqrt{\frac{\sigma_1}{\rho_0 (1 + e_1)}}$$

Another scheme of the motion of the string is possible. When transverse motion is present, the kink point moves with velocity

$$b_* = \sqrt{\frac{\sigma_1}{\rho_0 (1 + e_1)}}$$

We can therefore attain a velocity at which a point at the cone surface corresponding to the given kink point moves with velocity greater than b_*, i.e. $Q > 0$. Now according to (3.1) and (3.2) the condition $Q \neq 0$ is possible only when $\varphi = \pi/2$. But the first integral of (2.11) shows that at the extreme generatrix where $\theta = \pi/2$, $\varphi = (\pi/2) \sin \alpha$, i.e. φ undergoes a discontinuity. All this implies that another scheme of motion is possible: the region of purely longitudinal motions is adjacent to the cone, a kink forms in the string and its segment adheres to the extreme generatrix $(\theta = \pi/2)$ (wedge effect), and a second kink is formed and the string follows the surface of the cone, with the form of the string determined by the first integral of (2.11) (Fig.3). The latter scheme is realized at large velocities of the shock wave $v_0 \operatorname{tg} \alpha > b_*$.

4. Conditions at the strong shock waves. Let us write the equations of conservation of momentum and conditions of continuity of the displacements at the strong shock waves, in accordance with the wave scheme shown in Fig.3 which is realized when $v_0 \operatorname{tg} \alpha > b_*$. The conditions near the first kink A yield

$$\begin{gathered}
\rho_0 (b - w_t)(v - v_0 \cos \alpha - w_t \sin \alpha) = (\sigma_1 \sin \alpha - \sigma_2)(1 + e_1) \\
\rho_0 (b - w_t)(v_0 \sin \alpha - w_t \sin \alpha) = (Q + \sigma_1 \cos \alpha)(1 + e_1) \\
\frac{v_0}{\cos \alpha} - v = \frac{b - w_t}{1 + e_1} (1 + e_2), \quad b = v_0 \operatorname{tg} \alpha \\
w_t = - \psi (e_1), \quad \psi (e_1) = \int_{e_0}^{e_1} \sqrt{\frac{1}{\rho_0} \frac{d\sigma_1}{de_1}}\, de_1
\end{gathered} \tag{4.1}$$

where σ_2 and e_2 are the stress and deformation in the segment AB. Near the second kink B we have the following conditions:

$$\begin{gathered}
\rho_0 (c - v)\, r\theta^{\cdot} = -\sigma \cos \varphi \,(1 + e_2) \\
\rho_0 (c - v) \left(\frac{r}{\sin \alpha} - v\right) = (\sigma_2 - \sigma \sin \varphi)(1 + e_2) \\
r\theta^{\cdot} = -\frac{c - v}{1 + e_2} (1 + e) \cos \varphi, \quad \frac{r^{\cdot}}{\sin \alpha} - c = -\frac{c - v}{1 + e_2} (1 + e) \sin \varphi
\end{gathered} \tag{4.2}$$

It can be found from the solutions, that

$$r\theta^{\cdot} = -c \sin \varphi \cos \varphi \tag{4.3}$$

We note that if $r^{\cdot}/\sin \alpha$ is found from the formulas (2.11) and (2.7), then the expressions obtained will follow from the two last equations of (4.2).

Remembering that near the point B we have $\varphi = (\pi/2) \sin \alpha$, we can find the unknowns b, w_t, v, e, e_1, e_2, Q, c, $r\theta^{\cdot}$, $r^{\cdot}/\sin \alpha$ from the system of equations (4.1)—(4.3) provided that the relationship $\sigma = \sigma (e)$ is given. Here v is the velocity of the string particles in the segment AB and

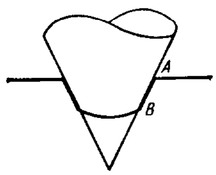

Fig.3

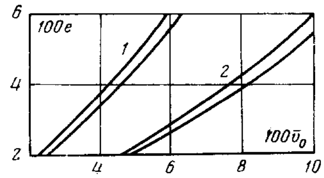

Fig.4

c is the velocity of the strong shock wave B. It can be shown that the system (4.2), (4.3) admits the solution $e = e_2$.

Figure 4 uses the Prandtl computational method for $e_s = 0.002\,(a_1/a_0)^2 = 0.05$ to depict graphs showing the dependence of the deformation e on the dimensionless velocity of the shock wave $\bar{v}_0 = v_0/a_0$. Here a_0 is the elastic wave velocity, a_1 is the plastic wave velocity and e_0 is the initial deformation of the string. The solid lines correspond to $e_0 = 0$ and the dashed lines to $e_0 = 0.001$, the curves *1* correspond to $\alpha = 60°$ and curves *2* to $\alpha = 75°$. We have $R = (c/v_0)\sin\alpha$ at the wave B and, according to the formula (2.7), we obtain

$$C_2 = \frac{c}{v_0}\sin\alpha\cos\left(\frac{\pi}{2}\sin\alpha\right)$$

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